"Life always has a fat tail"
— E. Fama (Econ Nobel)

Levy Flights – Systematics

(Follows Zaslavsky)

→ Have introduced Levy Flight, L-stable distribution. Here discuss systematically

→ Keep in mind:

- Levy distribution 'more general' than Gaussian
- L-stable distributions are Limit Distribution
- Only L-stable dist. of finite variance is Gaussian

→ Levy Flights → generalization of random walks to step pdfs with wild randomness

→ Levy Distributions (Generalizing CLT, Limiting Dist.)

General ideas

$P(x) \equiv$ pdf of random variable X

$\int_{-\infty}^{\infty} dx\, P(x) = 1$ → normalizability

with
Characteristic Fctn:

→ defines moments.

$$P(q) = \int_{-\infty}^{+\infty} dx\, e^{iqx} P(x)$$

so $\langle x^2 \rangle = \lim_{q \to 0} \left[i^2 \left(\frac{\partial}{\partial q} \right)^2 P(q) \right]$

etc.

"Stable Distribution":

x_1, x_2 and linear combos $cX_3 = c_1X_1 + c_2X_2$ (*)

P is stable if all X_1, X_2, X_3 distr. according $P(x_i)$

i.e. Gaussian – CLT.

Another class – Levy (1937)
using add. (*)

$$P(x_3)dx_3 = P(x_1)P(x_2)\,\delta\left(x_3 - \frac{c_1x_1 + c_2x_2}{c}\right)dx_1 dx_2$$

so, generators must satisfy, from *,

$c_1 x_1 + c_2 x_2 = c x$

id dist — id dist

$$\boxed{P(c\xi) = P(c_1\xi)\,P(c_2\xi)}$$

so

$$\boxed{\ln P(c\xi) = \ln P(c_1\xi) + \ln P(c_2\xi)}$$

These have solution:

$$\ln P_\alpha(c\xi) = (cq)^\alpha = c^\alpha e^{-i\frac{\pi}{2}\alpha(1-\mathrm{sign}\,q)}|\xi|^\alpha$$

where $\left(\frac{c_1}{c}\right)^\alpha + \left(\frac{c_2}{c}\right)^\alpha = 1$.

with arbitrary α.

so $P(x)$ with characteristic function

$$P_\alpha(q) = \exp\left[-c|q|^\alpha\right]$$

$\alpha = 2$ Normal

is Levy distribution with Levy index α.

$\boxed{0 < \alpha \le 2}$ → guarantees $P(x) > 0$.

and $\alpha = 2 \Rightarrow$ Gaussian (FT of G is G).

Observe:

$\Rightarrow \alpha = 2 \Rightarrow$ Gaussian

$\Rightarrow \alpha = 1 \Rightarrow$ Cauchy

$$P_1 = c/\pi \left[1/(x^2 + c^2) \right]$$

$\Rightarrow$ large $|x|$, $\quad 0 < \alpha < 2$

$$P_\alpha(x) \sim 1/|x|^{\alpha+1} \quad \Rightarrow \text{power law}$$

obviously has fat tail for $\alpha < 2$

so

$\Rightarrow \langle x^m \rangle$ diverge for $m > \alpha$

i.e. $\boxed{\alpha < 2 \Rightarrow \text{Variance diverges} \quad \text{can't construct F-P}}$

Now consider Levy process

→ Levy Process (analogue/generalization of diffusion)

- time dependent process
- has Levy distribution at infinitesimal time Δt.

Consider transition prob:

Chapman-Kolmog. eqn.

$$P(x_0, t_0 | x_N, t_N) = \int dx_1 \int dx_2 \dots \int dx_{N-1} \Big[P(x_0, t_0; x_1, t_1) * P(x_1, t_1; x_2, t_2) \dots P(x_{N-1}, t_{N-1}; x_N, t_N) \Big]$$

$$\int_0^N$$

$$t_{j+1} - t_j = \Delta t$$

$$N \gg 1$$

$$t_N - t_0 = N \Delta t$$

and assume process is uniform in space time (if not?);

$$P(x_j, t_j; x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j; t_{j+1} - t_j) = P(x_{j+1} - x_j, \Delta t)$$

so

$$P(x_N - x_0; N\Delta t) = \int dy_1 \dots dy_N \, P(y_1, \Delta t) \dots P(y_N, \Delta t)$$

Now, characteristic

$$P(q) = \int dy_j\, e^{iqy_j} p(y_j, \Delta t) \quad \rightarrow \text{step}$$

$$P_N(q) = \int dy^N\, e^{iqy^N} P(y^N, N\Delta t)$$

$$y^N = \sum_1^N y_j \quad = y_N - y_0$$

(cumulative)

$\Rightarrow$

$$\boxed{P_N(q) = [P(q)]^N}$$

$P(q) = P(q|\alpha, c)$ so

$$\begin{cases} P(q) = P_\alpha(q, \Delta c) \\ P_N(q) = P_\alpha(q, c_N) \end{cases}$$

above consistent if :

$$c_N = N\Delta c = N\Delta t \frac{\Delta c}{\Delta t} \equiv cN\Delta t = ct$$

so

$P_N \to$ $$P_\alpha(\xi, ct) = \exp\left[-c N \Delta t |\xi|^\alpha\right]$$

so

$$P_\alpha(\xi, t) = \exp\left(-ct|\xi|^\alpha\right)$$

characteristic fctn of Levy Process

$$P_\alpha(\xi, t) = \exp\left(-ct|\xi|^\alpha\right)$$

2 param: c, α. $\alpha < 2$ $\left\{ \begin{array}{l} \alpha = 2 \\ \text{Diffn.} \end{array} \right.$

so finally, $$P_\alpha(x, t) = \int d\xi\, e^{i\xi x} e^{-ct|\xi|^\alpha}$$

$$\underset{x \to \infty}{\sim} t / |x|^{\alpha+1}$$

and once again see:

$\underline{\langle x^2 \rangle \to \infty}$ for $\underline{\alpha < 2, \text{ at any } t,}$

divergent variance.

see Zaslavsky, Chapt. IV for mathematical extensions.

Intro to Anomalous Diffusion

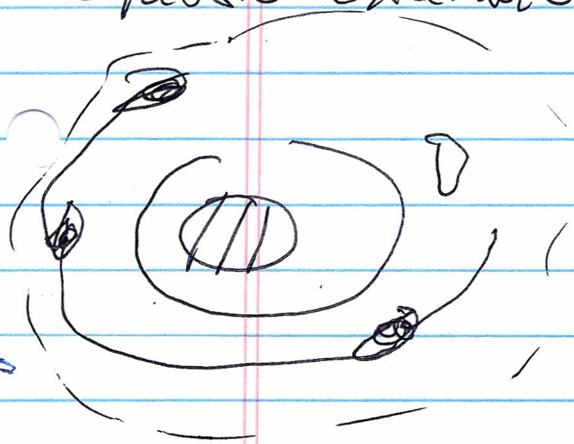
Anomalous diffusion:

$\langle \delta r^2 \rangle \sim t^\gamma$, instead $\langle \delta r^2 \rangle \sim t^{1/2}$

$\gamma > 1 \Rightarrow$ super-diffusive ($\gamma = 2$ - ballistic)
$\gamma < 1 \Rightarrow$ sub-diffusive

obv. $\gamma \leftrightarrow H$
connection

Classic example: Weeks, Swinney exp.



rotating tank $\Rightarrow$
eddys

tracer particle:
- trapped many rotation times in eddys
- occasional large flights

Anomalous diffusion can manifest other oddities $\Rightarrow$ up-gradient transport (I)

Phlm: How derive kinetic equation?

Review F-P:
- Δt clock, pdf Δx
- $\frac{\partial}{\partial x} \cdot D = \frac{\Delta x}{\Delta t}$

Now, how represent anomalous diffusion ?

- → CTRW (Continuous Time Random Walk)
- → Fractional kinetics * (main emphasis)

→ CTRW

- to represent, allow large step

i.e. $P(\Delta z) \sim 1/\Delta z^{1+\alpha}$

but $\alpha < 2$ precludes use of F-P expansion.

i.e. can't simply use fat tail distribution

- CTRW ⇒ accomodate flights / sticking in random walk model

→ release time from dummy role ⇒ allow evolve dynamically, as walker position does.

→ allow large (divergent) steps in space to be accomplished in ~~l~~arge step time so variance finite.

→ CTRW

→ position of n^{th} step

$r_n = \underline{r_0} + \Delta r_1 + \ldots \underline{\Delta r_{n-1}} + \underline{\Delta r_n}$

$t_n = t_0 + \Delta t_1 + \Delta t_2 \ldots \Delta t_n$

↓

time of n^{th} step

$\boxed{\text{Need Pdf}(\Delta t_i)}$

2 approaches to CTRW (see Klafter review)

→ waiting model:

- steps in position, time independent
- need specify 2 probabilities $\begin{cases} \Delta r \\ \Delta t \end{cases}$

idea: particle waits Δt in position (sticking), then jumps Δr in no time.

→ velocity model

- Δt → traveling time of particle

i.e

$\Delta t = |\Delta r|/v$

↳ const v.

so

increments satisfy $\delta(\Delta t - |\Delta r|/v)\, p(\Delta r)$

→ general → specify joint prob

Now, to build up CTRW equations: extend Chapman-Kolmogorov Eqn,

for dist jump pts:

Chap-Kolmo

$$Q(z,t) = \int d\Delta z \int_0^t d(\Delta t)\, Q(z-\Delta z, t-\Delta t)\, \overset{\text{step}}{p(\Delta z, \Delta t)}$$

+ i.c. + so.

so $P \equiv$ probability of walker to be at position z at time t

(1) in waiting model: factorize, need specify

$$\boxed{p(\Delta z, \Delta t) = p(\Delta z)\, p(\Delta t)}$$

$$P_W(z,t) = \int_0^t d(\Delta t)\, Q(z, t-\Delta t)\, \Phi_W(\Delta t)$$

Q

$$\Phi_w(\Delta t) = \int_{\Delta t}^{\infty} dt' \, \rho_{\Delta t}(t')$$

prob. to wait at least Δt.

(2) — velocity model:

$$\rho(\Delta z, \Delta t) = \delta(\Delta t - |\Delta z|/v)\, \rho(\Delta z)$$

$$f_v(z,t) = \int_{-vt}^{vt} d(\Delta z) \int_0^t d(\Delta t)\, Q(z-\Delta z, t-\Delta t)\, \Phi_v(\Delta z, \Delta t)$$

with:

$$\Phi_v = \tfrac{1}{2}\delta(|\Delta z| - v\Delta t) \int_{|\Delta z|}^{\infty} dz' \int_{\Delta t}^{\infty} dt' \, \delta(t' - |z'|/v)\, p(z')$$

prob. to make step of at least length $|\Delta z|$ and duration (Δt)

key question → what does the mess look like??

CTRW "diffusion"

→ depends on distribution of step increments, in both time, space — contrast F-P Eqn.

→ small steps both recovers "normal" diffusion

but

- { small spatial steps
  { Levy distributed (long) waiting times

⇒ subdiffusion, sticking

- { small time steps
  { Levy distributed (long) spatial steps

⇒ superdiffusion

→ CTRW eqns. non-local, non-Markovian in space time

C.C.

$$D\partial_x^2 P \to \partial_x\left\{\int dx' \int dt' \; * \; K(x-x', t-t')\, \partial_{x'} P(x', t')\right\}$$

scale of kernel critical → again approach via factorization

How solve this mess? → leave long tail process un-expanded on chap fil me

→ Fourier-Laplace Transform — i.e. work with generating function.

Recall: generating fctn.

$$P(k) = \int e^{-ik\Delta z} P(\Delta z)\, d\Delta z$$

so

$$i^n \partial_k^n P(k)\Big|_{k=0} = \int d\Delta z\, (\Delta z)^n P(\Delta z)$$

$$i^n \partial_k^n P(k)\Big|_{k\to 0} = \langle \Delta z^n \rangle$$

For CTRW solution, concentrate on asymptotic, large $|z|$ regime $\Leftrightarrow$ low k

Make Taylor expansion:

$$p(k) = p(0) + k\,\partial_k p(0) + \frac{k^2}{2}\partial_k^2 p(0)$$

$\Rightarrow$

$$p(k) = 1 - \cancel{i\langle \Delta z\rangle} k - \frac{1}{2}\langle(\Delta z)^2\rangle k^2 + \dots$$

0

symmetry

$$p(k) = 1 - \frac{1}{2}\langle(\Delta z)^2\rangle k^2$$

Similarly, time distributions (which can be 1 sided)

$$p(s) = 1 - \langle \Delta t\rangle s + \dots$$

in analogy

Which brings us back to Levy,

$$p^{L\alpha}(k) = \exp\left[-a|k|^{\alpha}\right]$$

$$\cong 1 - a|k|^{\alpha} + \cdots\cdots$$

and proceed following posted materials.

and proceed a/q l refs.

Summary

→ CTRW:

aka FP

- dist. Δt, Δz
  new → stickiness
- works directly with integral eqn
  chap - kolmog
- factorizes $p(\Delta z, \Delta t)$
- needs fat tail $p(\Delta t)$ dist.